This exam consists of five problems. Each problems is worth 20 points. Try to solve all problems. Your solutions must include intermediate steps and explanations. You must obtain at least 50 points from the exam in order to pass the course.

You are allowed to bring a hand-written A4-sized cheat sheet. You are allowed to write on both sides. Mark the hand-written collection with your name and return it with the exam paper. You are also allowed to bring a calculator. **Handbooks such as MAOL are not allowed.**

Problem 1. Consider a floating-point system that represents numbers as

$$x = (-1)^{b_4} \cdot f \cdot 2^{e-2},$$

where $b_4 \in \{0,1\}$ determines the sign and $f = (0.b_1b_0)_2$ ja $e = (b_3b_2)_2$ are the binary representations of the mantissa and exponent. Determine and write in decimal form the smallest positive number that can be represented using the above floating-point system.

Problem 2. Let $L(x)$ be the linear approximation (i.e., Taylor polynomial of degree one) to the function $f(x) = (x+1)x^2$ formed at the point $x_0 = 0$. Form $L(x)$ and use $L(1)$ to approximate $f(1)$. Compute

(a) error,
(b) absolute error, and
(c) relative error

of this approximation.

Problem 3. Consider the initial value problem

$$y' = xy, \quad y(0) = 2.$$

(a) Use Euler's method with two steps to solve this initial value problem at $x = 4$.

(b) Use the midpoint method with two steps to solve this initial value problem at $x = 4$.

Problem 4. The following four iterations are used to solve numerical problems:

$$y_{k+1} = y_k + hf(t_k, y_k), \quad \text{(a)}$$
$$x_{k+1} = x_k - \eta f'(x_k), \quad \text{(b)}$$
$$x_{k+1} = f(x_k), \quad \text{(c)}$$
$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \quad \text{(d)}$$

Here $\eta, h > 0$. Describe and give the mathematical form of the problem that each of the iterations is designed to solve numerically.

Problem 5. Let $Q_n(f)$ stand for the approximation of Simpson's rule with n sub-intervals to the integral $\int_a^b f(x)\,\mathrm{d}x$. If f is four times differentiable on $[a,b]$, the error of Simpson's rule satisfies

$$\int_a^b f(x)\,\mathrm{d}x - Q_n(f) = -\frac{(b-a)^5}{180n^4} f^{(4)}(\xi)$$

for some $\xi \in (a,b)$.

(a) Show that Simpson's rule is exact for all polynomials of degree at most 3.

(b) Let us set $b = 1$, $a = 0$ and $n = 1$. Give an example of a function f such that

$$\int_a^b f(x)\,\mathrm{d}x - Q_n(f) = 1.$$