

Final Examination

Course: HDD4020 Engineering Mathematics II

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Date: 17.12.2025 Total marks: 60

Note: Calculators are **not** allowed. Solutions must include the applied formulas, substitutions, and intermediate steps. Submitting only the final answer is not acceptable.

There are **SIX** questions in the paper. Answer **ALL** questions. Questions carry equal marks. Throughout the paper, $i = \sqrt{-1}$, and z is a complex number, except in Question 2(b) and Question 3(b), where it is real.

Question 1

(a) Find the rectangular coordinates of the point whose spherical coordinates are

$$\left[4, \frac{\pi}{3}, \frac{\pi}{6}\right].$$

(b) Find the length of the parametric curve $x = \sin^3 t$, $y = \cos^3 t$, $(0 \leq t \leq 2\pi)$.

Question 2

(a) Find the eigenvalues of $A = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 5 & -1 \\ 4 & 3 & 1 \end{pmatrix}$.

(b) Find the spherical equation of the Cartesian equation $x^2 + y^2 + (z + 2)^2 = 4$.

(c) Find the parametric equations of the line that passes through $(1, 2, 3)$ and $(0, 1, -1)$.

Question 3

(a) Find $\lim_{z \rightarrow -1+i} \frac{z^2 + 2i}{z^2 + 2z + 2}$.

(b) Solve the linear system
$$\begin{cases} x - y + z = 2 \\ x + y + 2z = 4 \\ 2x + 4y + 5z = 10 \end{cases}.$$

Question 4

- (a) The curve is given by the following parametric equations

$$x = t^6 - 6, \quad y = t^3 + 1.$$

Find the equation of the normal line to the curve at the point where $t = 0$.

- (b) Compute $(-1 - i)^8$. Write your answer in the Cartesian form $a + bi$.

Question 5

- (a) Let $\mathbf{A}$ be the matrix representing the triangle with vertices $P(2,1)$, $Q(6,1)$, and $R(6,3)$, and let $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Determine the effect of the transformation $\mathbf{CBA}$ on the image of the triangle.

- (b) Find the value of a such that the four points $(0, 1, -2)$, $(-2, 3, 0)$, $(1, -1, 1)$, and $(2, 1, a)$ lie in the same plane.

- (c) Find $\sqrt[3]{i}$. Write your answer in the Cartesian form $a + bi$.

Question 6

- (a) Write $f(z) = (z + 1)^2$ in the form $f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$.

- (b) Find the image of the line $y = 1$ under the transformation $w = f(z) = z^{-1}$.

List of Potentially Useful Formulas

$$\sin \frac{\pi}{3} = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}, \sin \frac{\pi}{6} = \cos \frac{\pi}{3} = \frac{1}{2}$$

$$\sin x \cos y = \frac{1}{2} [\sin(x+y) + \sin(x-y)]$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\frac{d}{dx} x^n = nx^{n-1}$$

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$(f(g(x)))' = f'(g(x))g'(x)$$

$$\begin{cases} x = f(t_0) + f'(t_0)(t - t_0) \\ y = g(t_0) + g'(t_0)(t - t_0) \end{cases}$$

$$y - y_0 = m(x - x_0)$$

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$A = \int_{t_1}^{t_2} 2\pi|x| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$A = \int_{t_1}^{t_2} 2\pi|y| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

$$\rho^2 = x^2 + y^2 + z^2$$

$$|A - \lambda I| = 0$$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$- a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\hat{u} = \frac{\mathbf{u}}{|\mathbf{u}|}$$

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = d$$

$$s = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

$$z = r(\cos\theta + i\sin\theta)$$

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

$$|z|^{1/n} \left(\cos \frac{\theta + 2\pi k}{n} + i \sin \frac{\theta + 2\pi k}{n} \right),$$

$$k = 0, 1, \dots, n-1.$$

$$\overline{x + yi} = x - yi$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$