

BM20A8401 Partial Differential Equations
Exam 09.09.2026

Exam Instructions

- Only stationery (pen and empty paper) is allowed in the exam. (*Sama Suomeksi: Ainoastaan tavalliset kirjoitusvälineet ovat sallittuja tentissä.*)
- Answer to **all five problems**; each is worth **6 marks**; the total marks for the exam is **30**.

1. Instructions:

- For each statement below, determine whether it is **True** or **False**.
- You will receive **1 mark** for each correct answer, **-1 mark** for each incorrect answer and **0 marks** for each unanswered statement.
- However, the total marks for this question cannot be negative, meaning that even if you accumulate negative points, the final score for this question will be at least **0 marks**.

Statements:

- (i) The function $d : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2|$$

is a metric on $\mathbb{R}^2$.

- (ii) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \exp(-x^2).$$

Then the support of f is the empty set.

- (iii) Let $\{K_i\}_{i=1}^m$ be a finite collection of compact subsets of $\mathbb{R}^n$. Then

$$\bigcup_{i=1}^m K_i$$

is compact.

- (iv) The type of a second-order linear partial differential equation is invariant under a non-singular change of coordinates. That is, its classification as elliptic, parabolic, or hyperbolic does not depend on the coordinate system used.

- (v) Let $n > 2$ and define

$$u : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$$

by

$$u(x_1, \dots, x_n) = (x_1^2 + \dots + x_n^2)^{1+\frac{n}{2}}.$$

Then u is harmonic in $\mathbb{R}^n \setminus \{0\}$.

- (vi) Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $f : \Omega \rightarrow \mathbb{R}$ be real-analytic in Ω . Then f is harmonic in Ω .

2. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by $f(x, y) = \frac{1}{1+x+y}$. Determine the Taylor polynomial of degree 2 of f centered at $(0, 0)$.

3. Use the method of characteristics to find a solution to the initial value problem

$$u_y = xu u_x,$$

subject to the initial condition

$$u(x, 0) = x.$$

4. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f : \Omega \rightarrow \mathbb{R}$ be a smooth function.

- (a) Define what it means for f to be harmonic in Ω .
- (b) State the mean value property for harmonic functions over balls contained in Ω .
- (c) Investigate whether the following statements are true or false. Justify your answer with a proof or a counterexample.
 - (i) Let $\Omega \subset \mathbb{R}^n$ be an open and connected set. Suppose $u, v \in C(\overline{\Omega}) \cap C^2(\Omega)$ are harmonic in Ω and satisfy

$$u = v \quad \text{on } \partial\Omega.$$

Then

$$u \equiv v \quad \text{in } \Omega.$$

- (ii) Let $\Omega \subset \mathbb{R}^n$ be connected and let $u : \Omega \rightarrow \mathbb{C}$ be harmonic. If $|u|$ attains a minimum at a point in Ω , then u is constant in Ω .

5. Let $\{F_i\}_{i \in I}$ be a collection of closed subsets of $\mathbb{R}^n$.

- (a) Suppose that I is finite. Using the fact that the complement of a closed set is open, show that

$$\bigcup_{i \in I} F_i$$

is closed in $\mathbb{R}^n$. Clearly indicate where the assumption that I is finite is used in your argument.

- (b) Is the finiteness assumption in part (a) necessary? In other words, if I is an arbitrary index set and each F_i is closed, must

$$\bigcup_{i \in I} F_i$$

be closed? Justify your answer by giving a proof or a counterexample.