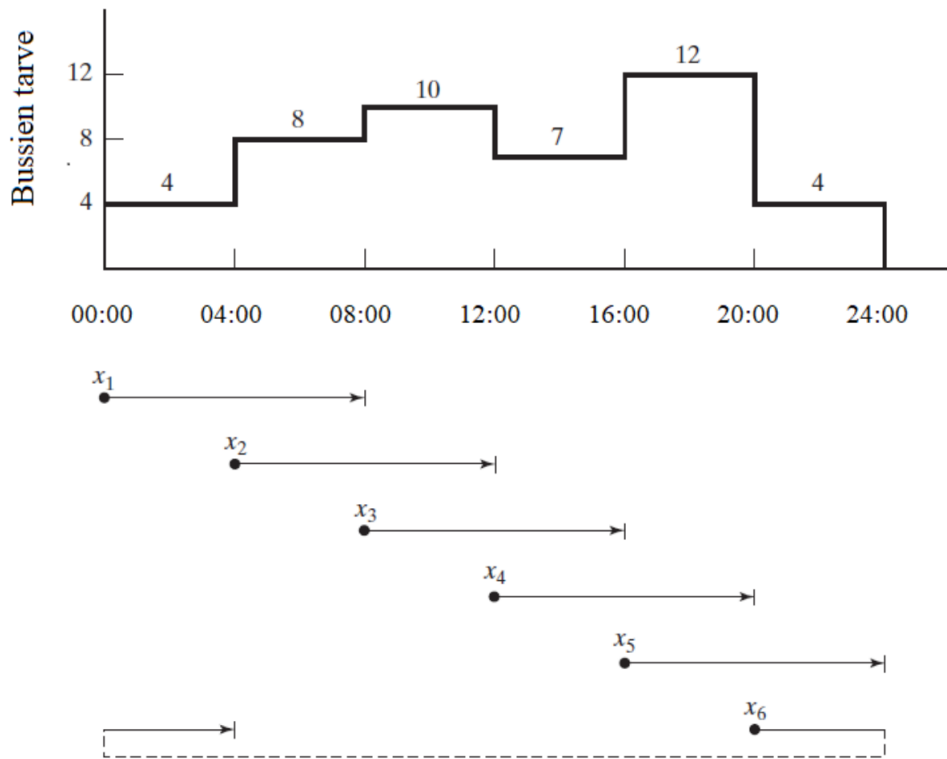


BM20A8201 Optimization

Exam 21.09.2026

Exam Instructions

- Only stationery (pen and empty paper) is allowed in the exam. (*Sama Suomeksi: Ainoastaan tavalliset kirjoitusvälineet ovat sallittuja tentissä.*)
 - Answer to at most **five (5) problems**; each is worth **6 marks**; the total marks for the exam is **30**. If you answer to all six (6) problems, only the first five (5) will be evaluated.
1. Engineer of the City, Möttönen, has been assigned the task of reducing emissions from public transportation. To achieve this, she has collected data in Finnish to determine the minimum number of buses as a function of time (see the figure below). The variable x_i represents the number of buses available during the eight-hour interval i , each interval being predetermined. Formulate a linear optimization model that minimizes the total number of buses.



2. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x_1, x_2) := x_1^2 + 5x_2^2 + 2x_1x_2 - 10x_1 - x_2 - 200$.
- Compute the gradient $\nabla f(x_1, x_2)$ of the function in an arbitrary point $(x_1, x_2) \in \mathbb{R}^2$.
 - Find all critical points of the function f .
 - Compute the Hessian matrix $\text{Hess } f$ and find its eigenvalues.
 - Does the function have a global minimum? Does the function have a global maximum? Determine the extreme points if they exist. Justify your answers using the information about the definiteness of the Hessian matrix.

The exam continues on the next page!

3. Let us define in Matlab the following code lines:

```
B = [0, 0, 0, 0];
C = [55, 40, 20, 25];
h = -[80, 70, 95, 90];
X = [5, 7.5, 4, 4; 1, 1, 1, 1; 2, 2.5, 3.5, 3.5; 0.5, 0.5, 1, 0.5];
d = [450, 120, 280, 80];
[x,fval,exitflag,output] = linprog(h,X,d,[],[],B,C);
```

- (a) Write down the corresponding mathematical LP problem.
 - (b) Explain what are the resulting elements of the linprog algorithm: x, fval, exitflag, output. Only brief descriptions of 'exitflag' and 'output' are sufficient.
4. (a) Define Newton's algorithm for searching critical points of a multivariate function in a mathematical form. Define all used mathematical notation.
- (b) Let $f(x, y) = x^6 + 2x^2 + 20y^2 - 2x - 2y - 1$. Write a Matlab code, based on (a), that searches for a critical point of f using Newton's algorithm from the initial point $(x, y) = (1, 0)$ and with the tolerance $\epsilon = 10^{-5}$.
5. Let $U \subset \mathbb{R}^n$ be open and $f : U \rightarrow \mathbb{R}$ differentiable.
- (a) Write the definition of critical points of f .
 - (b) Write the definition of local extremum (minima and maxima) of f .
 - (c) Show that if $a \in U$ is a local extremum f , then a is a critical point of f .

6. Let

$$A := \begin{bmatrix} 1 & 2 & -1 & 4 \\ 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 2 \end{bmatrix}, \quad b := \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}, \quad c := \begin{bmatrix} 0 \\ 3 \\ 1 \\ 2 \end{bmatrix}.$$

- (a) Write the mathematical definition of a convex polyhedron.
- (b) Write the mathematical definition of a vertex of a convex polyhedron.
- (c) Find the vertices of

$$\mathcal{P} := \{x \in \mathbb{R}^4 : Ax = b, 0 \leq x\}.$$

You may use a theorem presented in the lecture notes.

- (d) Find the minimum value of the objective function $x \mapsto c^T x$ over $\mathcal{P}$. Give a point where this minimum is attained.