BM20A8000 Differential Equations — Exam 5.5.2026

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Only the following written materials are allowed in the exam: MAOL, BETA, Tekniikan taulukkokirja (Valtanen), Matematiikan taulukkokirja (Valtanen), and a handwritten A4-sized formula sheet. The formula sheet must not contain any example problems, and **it must be returned along with the exam paper**. Calculators are not allowed.

Each problem is worth 6 points. Write clearly, and **remember to justify your answers**.

1. Solve the following differential equations / initial value problems. On which intervals are the solutions defined?

(i) $y' + y^2 = 0$; (ii) $y''(x) - \cos(x) - 3 = 0, \quad \begin{cases} y(1) &= 1, \\ y'(1) &= 1 \end{cases}$

(3 p per item).

2. Are the following differential equations separable? Solve all separable equations by separation of variables.

(i) $y' = e^x(1+y)$; (ii) $y = e^x(1+y')$; (iii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y(x^3-1)+y}{x(y^3-1)+x}$

(2 p per item).

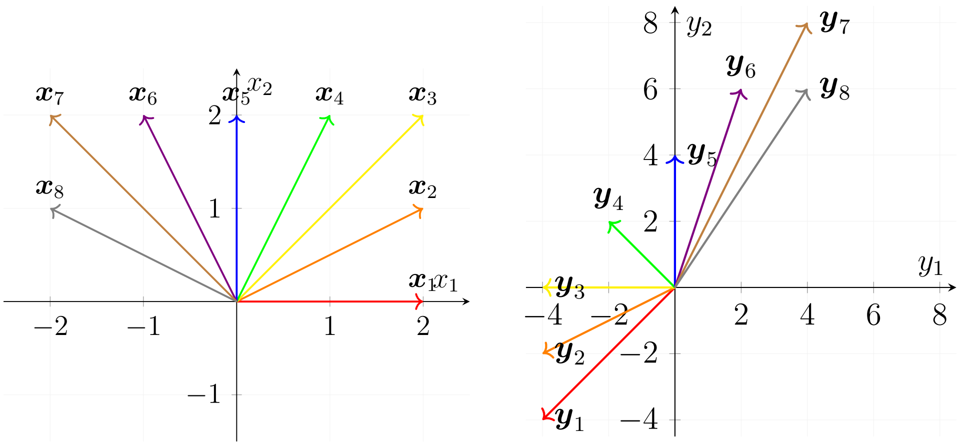
3.

(i) Determine the eigenvalues of the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

and all eigenvectors corresponding to one of the eigenvalues (of your chocie). (3 p)

(ii) In the left-hand figure below there are vectors $\boldsymbol{x}_i$, and in the right-hand figure there are vectors $\boldsymbol{y}_i = \mathbb{A}\boldsymbol{x}_i$, that is, the right-hand side vectors are obtained by multiplying the left-hand side vectors by a 2×2 matrix $\mathbb{A}$.
Vectors of the same color correspond to each other. Based on the figures, determine the eigenvalues and eigenvectors of the matrix $\mathbb{A}$. (3 p)



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4. A small lake (volume $V = 2\,000\,000\,\mathrm{m}^3$) receives water from a river whose flow rate is constant,

$$r = 100\,000\,\frac{\mathrm{m}^3}{\mathrm{month}}.$$

Water leaves the lake at the same rate, so the volume remains constant. The salt in the lake is assumed to be uniformly mixed with the water.

The salt concentration in the river varies seasonally, and is given by

$$c_{\mathrm{in}}(t) = 4 + 2\sin(\pi t/6) \qquad \left(\frac{\mathrm{mg}}{\mathrm{l}}\right),$$

where t denotes time measured in months. At time $t = 0$, there is $4\,000\,\mathrm{kg}$ of salt in the lake.

Let $Q(t)$ denote the amount of salt (in grams) in the lake at time t (months).

(i) Form an initial value problem that describes the quantity $Q(t)$; (2 p)

(ii) Solve the initial value problem. *Hint:* $\int e^{at}\sin(bt)\mathrm{d}t = \frac{e^{at}(a\sin(bt) - b\cos(bt))}{a^2+b^2} + C$. (4 p)

5. Let $\sigma \in \mathbb{R}$. Consider the boundary value problem

$$y''(t) - \sigma y(t) = 0, \qquad 0 \le t \le 1, \qquad \begin{cases} y(0) &= 0, \\ y(1) &= 0. \end{cases}$$

Determine all values of the parameter σ for which this boundary value problem has a multi-valued solution.