Course: **BM20A4703** Partial Differential Equations with Applications
Exam date: **May 04, 2026**. Time: **3 hours**
(No computers or documents are allowed)

(The exam consists of 5 problems. Each problem is worth 20 marks.)

Problem 1

(20 marks) Solve the following PDE using the method of separation of variables

$$u_{tt} - u_{xx} = 0, \quad x \in [0, \pi],\ t > 0,$$

with $u(x,0) = u(x,0) = x(\pi - x),\ u_t(x,0) = 0,\ u(0,t) = u(\pi,t) = 0.$

Problem 2

(20 marks) Let $U = B_1(0)$ be the open unit ball in $\mathbb{R}^n, n > 1$, and define $u : U \to \mathbb{R}$ by

$$u(x) = \log\log\left(1 + \frac{1}{|x|}\right).$$

Verify that u belongs to $W^{1,n}(U)$.
Hint: Using polar coordinates, show that both u and ∇u belong to $L^n(U)$.

Problem 3

Derive the conservation of mass starting from the balance

$$\frac{d}{dt}\int_{\Omega(t)} \rho(\vec{x},t)\ d\Omega = 0.$$

Hints: Use the Reynolds transport theorem

$$\frac{d}{dt}\int_{\Omega(t)} f\ d\Omega = \int_{\Omega(t)} \frac{\partial f}{\partial t}\ d\Omega + \int_{\partial\Omega(t)} f\mathbf{u}\cdot\mathbf{n}dS$$

and the Green's formula

$$\int_{\partial\Omega} f\mathbf{u}\cdot\mathbf{n}dS = \int_{\Omega} \mathbf{u}\cdot\nabla f\ d\Omega + \int_{\Omega} f\nabla\cdot\mathbf{u}\ d\Omega.$$

Problem 4

(20 marks) The axi-symmetric Navier-Stokes equations simplify to the following form for a fully-developed flow in a tube:

$$-\frac{1}{r}\frac{\partial}{\partial r}\left[r\mu\frac{\partial u_z}{\partial r}\right] + \frac{\partial p}{\partial z} = 0, u_z(R) = 0, \frac{\partial u_z}{\partial r}|_{r=0} = 0,$$

when the gravity can be neglected. Let us assume that the viscosity follows the Power-law model:

$$\mu = k\left(\frac{\partial u_z}{\partial r}\right)^{n-1}.$$

Solve the fully-developed velocity profile $u_z(r)$ for a given pressure gradient in the z direction: $\frac{\partial p}{\partial z} = -\Delta p/L$, that is, a known pressure difference for a length L. Explain also how the maximum velocity in the middle of the tube behaves as a function of the Power-law index n.

Problem 5

(20 marks)

- The Navier-Stokes equations are the basis for all fluid dynamics. Explain what kind of simplified equations can be used depending on Reynolds number or when the flow is irrotational.
- Explain what turbulence means and how turbulent flows can be mathematically modelled by using partial differential equations.