BM20A8501 Probabilistic Simulation — Exam 10.3.2026

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Each question is worth 6 points. Please write clearly, and **remember to justify your answers**. You may answer either in English or in Finnish.

1. Consider a vector

$$[p_1, p_2, \cdots, p_N]$$

of positive numbers p_i such that

$$\sum_{i=1}^{N} p_i = 1.$$

Describe a mathematical method to transform a uniform random variable $R \sim U([0,1])$ into a discrete random variable X such that

$$\mathbb{P}\left(X = A_i\right) = p_i \quad \text{for all} \quad i \in \{1, 2, \cdots, N\}$$

for some given labels $\{A_1, A_2, \cdots, A_N\}$.

2. Suppose you have obtained a data set of real numbers, whose underlying statistical distribution is unknown, but you can reasonably expect the distribution to be continuous and the samples to be independent of each other. Describe a method to generate new random numbers that approximately follow the unknown distribution. How can you make use of the a priori knowledge that the distribution is continuous?

3. Consider the following exotic dice game.

(i) A fair dice with 16 sides (numbered $1, 2, 3, \cdots, 15, 16$ respectively) is thrown twice;
(ii) The product of the two resulting numbers is formed;
(iii) This product is divided by 10 and rounded down to the nearest integer.

Present a pseudocode algorithm that simulates an approximation of the finite-dimensional statistics of the dice game. You can assume access to standard mathematical operators, including the floor and ceiling operators.

4. Describe the agent-based method of dynamical simulation (ABM). Discuss

(a) its benefits;
(b) possible limitations to using it.

(Continued on the next page)

5. Consider the following continuous-time model: a particle p on the Euclidean plane $\mathbb{R}^2$ spawns at the origin $(0,0)$, and starts moving with velocity 1 in a randomly chosen direction. At random time points, p changes its direction to another randomly chosen one, where it continues to move with velocity 1.

(a) Present a pseudocode algorithm to generate a random direction for the particle p. This should be a uniformly distributed point on the unit circle, i.e. a random vector $(x,y) \in \mathbb{R}^2$ satisfying $x^2 + y^2 = 1$ with probability 1. (2p)

(b) Present a pseudocode time-discretized implementation of the moving particle model with discretization interval $\delta > 0$ and final time $T > \delta$, with the assumption that the time intervals between the random direction changes in the continuous model are distributed independently as Exponential(1).
You can assume access to the random directions generated in part (a), and the exponential distribution. Your implementation can have e.g. $\lceil T/\delta \rceil$ time steps, so the final time in the discretized model does not have to exactly match T. In this case, the output of the algorithm should be a $\lceil T/\delta \rceil \times 2$ array, where each row describes the coordinates of the particle p at a time-discreziation point. (4p)