

- There are 5 problems in this exam. Each problem is worth 20 points.
- You have 3 hours to complete the exam.
- All calculators are allowed.
- **You must include intermediate steps in your solutions.**

**Problem 1.** Let  $f: (a, b) \rightarrow \mathbb{R}$  be a source term. Poisson’s equation on the interval  $(a, b)$  with zero Dirichlet boundary conditions is

$$\begin{aligned} -u'' &= f \quad \text{in } (a, b), \\ u(a) &= u(b) = 0. \end{aligned}$$

Derive the weak form of Poisson’s equation.

**Problem 2.** Consider Poisson’s equation on  $(0, 1)$  with the source term  $f(x) = x^2$ . Let the uniform nodes

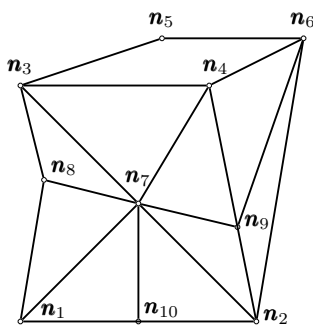
$$x_i = i/N \quad \text{for } i \in \{0, \dots, N\}$$

define a partition of the interval. Consider the finite element space of continuous piecewise linear functions with zero boundary conditions. The coefficients  $\beta$  of the finite element solution are solved from the linear system  $\mathbf{A}\beta = \mathbf{b}$ . Assemble (i.e., compute) the matrix  $\mathbf{A} \in \mathbb{R}^{(N-1) \times (N-1)}$  and vector  $\mathbf{b} \in \mathbb{R}^{N-1}$  for the hat basis functions.

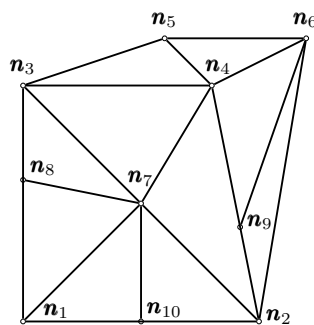
**Problem 3.** Let  $I = (0, r)$  for  $r > 0$  and  $v \in C_0^\infty(I)$ . Prove the Poincaré–Friedrichs inequality

$$\|v\|_{L^2(I)} \leq r \|v'\|_{L^2(I)}.$$

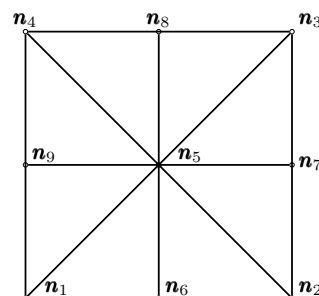
**Problem 4.** Which of the three meshes below are triangular and conforming? Justify your answers.



mesh 1



mesh 2



mesh 3

**Problem 5.** Let  $V$  be a Hilbert space and  $a: V \times V \rightarrow \mathbb{R}$  a bilinear form. One of the assumptions of the Lax–Milgram theorem is that  $a$  is *elliptic*, which is to say that there is  $\alpha > 0$  such that  $a(v, v) \geq \alpha \|v\|_V^2$  for all  $v \in V$ .

Let  $\Omega = (0, 1)^2 \subset \mathbb{R}^2$  and  $V = H_0^1(\Omega)$ .

(a) Show that the following bilinear form is **elliptic**:

$$a(u, v) = \int_{\Omega} \mathbf{K} \nabla u \cdot \nabla v \quad \text{for} \quad \mathbf{K} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}.$$

(b) Show that the following bilinear form is **not elliptic**:

$$a(u, v) = \int_{\Omega} \mathbf{K} \nabla u \cdot \nabla v \quad \text{for} \quad \mathbf{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

## Collection of potentially useful formulae

$$A_{ij} = \int_0^1 \varphi'_i \varphi'_j$$

$$A_{ij} = \int_{\Omega} \mathbf{K} \nabla \varphi_i \cdot \nabla \varphi_j$$

$$b_i = \int_{\Omega} f \varphi_i$$

$$a(u, v) = L(v)$$

$$\int_a^b uv' = u(b)v(b) - u(a)v(a) - \int_a^b u'v$$

$$\int_{\Omega} \nabla \cdot \mathbf{G} = \int_{\partial\Omega} \mathbf{G} \cdot \mathbf{n}$$

$$\nabla \cdot (\mathbf{F}\varphi) = \varphi \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla \varphi$$

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \quad \text{if} \quad u \in H_0^1(\Omega)$$

$$\langle u, v \rangle_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

$$u(x) = u(0) + \int_0^x u'(z) dz$$

$$\|u\|_{H^m(\Omega)} = \left( \sum_{|\alpha| \leq m} \|\partial_{\alpha} u\|_{L^2(\Omega)}^2 \right)^{1/2}$$

$$|u|_{H^m(\Omega)} = \left( \sum_{|\alpha|=m} \|\partial_{\alpha} u\|_{L^2(\Omega)}^2 \right)^{1/2}$$

$$\|u\|_{L^2(\partial\Omega)} \leq C \|u\|_{H^1(\Omega)}$$

$$\int_T f(\mathbf{x}) d\mathbf{x} = \int_{\hat{T}} f(F_T(\mathbf{x})) |\det \mathbf{A}_T| d\mathbf{x}, \quad \text{where} \quad F_T(\mathbf{x}) = \mathbf{A}_T \mathbf{x} + \mathbf{b}_T$$

$$\|u - u_h\|_E = \inf_{v_h \in V_h} \|u - v_h\|_E$$

$$\int_{\Omega} \varphi \nabla \cdot \mathbf{F} = - \int_{\Omega} \mathbf{F} \cdot \nabla \varphi + \int_{\partial\Omega} \varphi \mathbf{F} \cdot \mathbf{n}$$