BM20A4102 Vector Analysis, Fall 2025
Exam 24.3.2026

Writing instruments and a photo ID are permitted in the exam. Personal notes, lecture notes, course books, table books, phones, computers, and calculators are not allowed. Answer all five questions. Explain each of your answers carefully. Remember to write your name on all answer sheets.

Question 1

(a) Let $f : \mathbb{R}^n \to \mathbb{R}$. Give the definition of the gradient of the function f.

(b) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_3, x_2^2, \sin x_1)$. Compute the matrix representation of the derivative $Df(\pi/2, 1, 0)$.

(c) Let $D := [0,1] \times [0,1]$. Consider the mapping $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) := 2e^{x-y}$. Compute the value of the integral $\int_D g(x,y)\,dx\,dy$.

Explain each of your answers carefully.

Question 2

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) := e^{x_2^2 - x_1}$. Find a polynomial $p : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{|f(x) - p(x)|}{\|x\|^2} \to 0 \quad \text{as } x \to (0,0).$$

Explain your answer carefully.

Question 3

Consider the surface

$$S := \{(\cos\alpha, \sin\alpha, z) \in \mathbb{R}^3 : z \in (-1,1),\ \alpha \in (0,\pi)\}$$

with the parametrization

$$r : (0,\pi) \times (-1,1) \to \mathbb{R}^3, \quad r(\alpha, z) := (\cos\alpha, \sin\alpha, z).$$

Give the definition of the surface area of S using the given parametrization. Compute the exact value of the surface area.

Explain your answers carefully.

Question 4

Let

$$U := \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4,\ y > 0\}.$$

Compute the value of the integral

$$\int_U \|(x,y)\|\,dx\,dy.$$

Explain your answer carefully.

Question 5

Let $R > 0$. Derive the formula $L = 2\pi R$ for the circumference L of a circle of radius R.

Explain your answer carefully.

Formulas:
$v \cdot w = \sum_{j=1}^{n} v_j w_j$,
$v \times w = (v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1)$,
$\nabla f(x) = (\partial_1 f(x), \ldots, \partial_n f(x))$,
$f(a+h) = f(a) + \left[\sum_{l=1}^{k} \frac{1}{l!} \sum_{j_1=1}^{n} \cdots \sum_{j_l=1}^{n} h_{j_1} \cdots h_{j_l} \partial_{j_l} \cdots \partial_{j_1} f(a)\right] + \|h\|^k R(h)$,
$\mathrm{div}(F) = \sum_{j=1}^{n} \partial_j F_j$,
$\mathrm{curl}(F) = (\partial_2 F_3 - \partial_3 F_2,\ \partial_3 F_1 - \partial_1 F_3,\ \partial_1 F_2 - \partial_2 F_1)$,
$\Delta f = \mathrm{div}(\nabla f)$.